Synthetic Differential Geometry

Contents

1 Introduction

"In order to treat mathematically the decisive abstract general relations of physics, it is necessary that the mathematical world picture involve a cartesian-closed category $\mathcal{E}$ of smooth morphisms between smooth spaces."
F. William Lawvere in [Lawvere, 1980].

Synthetic Differential Geometry makes rigorous the notions of infinitesimal object, infinitesimal action and infinitesimal transformation which are often useful when thinking heuristically about Differential Geometry.

Instead of working in the category Man that has smooth manifolds as objects and smooth maps as arrows, in Synthetic Differential Geometry one works in a certain cartesian closed category[1] $\mathcal{E}$, which contains appropriately defined infinitesimal objects. The cartesian closed property is important because it means that certain analytic concepts can be replaced by standard category-theoretic constructions.

As an example of this, tangent vectors of an object M are defined as arrows from a certain infinitesimal object D to M. Using the fact that $\mathcal{E}$ is cartesian closed we define the tangent bundle of M to be M^D. Then the existence of the adjunctions

$$\frac{M \to M^D}{D \times M \to M}$$

and

$$\frac{D \times M \to M}{D \to M^M}$$

will allow us to identify rigorously (rather than just intuitively) the following:

(i) vector fields over M,

(ii) infinitesimal actions on M and

(iii) infinitesimal transformations of M.

The ability to easily switch between these different perspectives is used in the applications in Sections 3.4 and 3.5. In the former working with infinitesimal actions provides an alternative way of solving a particular differential equation and in the latter viewing vector fields as infinitesimal transformations will give a clear geometric meaning to the Lie bracket defined there.

However, we shall see (in Section 3.1) we cannot make rigorous these ideas in Set, the category of sets and functions (or in fact any other Boolean topos). The

[1] In fact in a *smooth topos* as defined in Section 3.2.

reason for this is that the fundamental axiom required for Synthetic Differential Geometry contradicts the Law of the Excluded Middle:

$$\top \vdash_{\vec{x}} (\phi \vee (\neg \phi))$$

(where we have assumed an appropriate deductive system and definition of a formula ϕ in the internal logic of a category). Therefore we must work in a category whose internal logic is intuitionistic. Section 2 will discuss the interpretation of logic in a category in more detail.

The approach taken in this essay will be to first give an axiomatic description of a *smooth topos* (see Section 3.2) in which there exist infinitesimal objects with the appropriate properties. The internal logic of this topos is intuitionistic. Then the axiomatic theory will be used in two applications (Sections 3.4 and 3.5). Finally Sections 4 and 5 will build up the theory needed to construct a model of a smooth topos. The existence of this model will justify our previous work.

The model we will build also satisfies the property of being *well-adapted* (which is explained at the beginning of Section 5). This property is an important one because it implies that results proved in this model of a smooth topos (and therefore any properties that follow from the axiomatic description of a smooth topos) may be interpreted as results about smooth manifolds.

2 Generalities in Categorical Logic

This section contains generalities in categorical logic that will be required in the rest of the essay.

It builds towards the definition of the syntactic category of an algebraic theory (Definition 2.13). This category will be used to define the k-algebras (for a ring k) and the C^∞-algebras that play a central rôle in the axiomatic theory and the model building.

The treatment of the theory in the first three subsections is similar to Chapter D1 of Volume 2 of [Johnstone, 2006]. [Awodey and Bauer, 2009] was also used, in particular for subsection 2.4. The definitions and notations in subsection 2.5 can also be found in [Kock, 2006].

2.1 Signatures

Definition 2.1. A *signature* Σ consists of:

- A finite set of *sorts* $X_1, X_2, ..., X_n$.
- A collection of *function symbols*. Each function symbol f is assigned a list of sorts $X_1, X_2, ..., X_k$ (with the X_i not necessarily distinct) as *arguments* and a single sort Y as *output*.
- A collection of *constants* each of which is assigned a sort.
- A collection of *relation symbols*. Each relation symbol R is assigned a list of sorts $X_1, X_2, ..., X_k$ (with the X_i not necessarily distinct and $k \neq 0$) as a *context*.

An interpretation I of a signature Σ in a category $\mathcal{C}$ with finite products is as follows:

- The interpretation of each sort is an object of the category. We will write the object interpreting X_i as X_i^I.
- The interpretation of a function symbol with arguments $X_1, X_2, ..., X_k$ and output Y is an arrow f^I

$$X_1^I \times X_2^I \times ... \times X_k^I \xrightarrow{f^I} Y^I \tag{2.1}$$

  in the category. We will abuse notation by specifying the arguments and output of a function symbol using a diagram like 2.1 but without the superscripts.

- The interpretation of a constant requires $\mathcal{C}$ to have a terminal object 1. The interpretation of a constant assigned the sort X_k is then an arrow:

$$1 \rightarrow X_k^I$$

of $\mathcal{C}$.

- The interpretation of a relation symbol with domain $X_1, X_2, ..., X_k$ is a subobject of $X_1^I \times X_2^I \times ... \times X_k^I$:

$$[(x_1, x_2, ..., x_k)|Rx_1x_2...x_k]^I \xrightarrow{R^I} X_1^I \times X_2^I \times ... \times X_k^I \tag{2.2}$$

At the moment $[(x_1, x_2, ..., x_k)|Rx_1x_2...x_k]^I$ is simply a symbol for the domain of the subobject. We will abuse notation by specifying the context of a relation symbol using a diagram like 2.2 but without the superscripts.

Example 2.2. The *signature of rings* Σ_{ring} has:

- A single sort k.
- The function symbols:

$$1 \underset{0}{\overset{1}{\rightrightarrows}} k \underset{\times}{\overset{+}{\leftleftarrows}} k \times k$$

- The relation symbol

$$[(k_1, k_2)|k_1 = k_2] \xrightarrow{=} k \times k$$

So an interpretation I of the signature of rings in a category is simply:

- An object k^I.
- Arrows:

$$1^I \underset{0^I}{\overset{1^I}{\rightrightarrows}} k^I \underset{\times^I}{\overset{+^I}{\leftleftarrows}} k^I \times k^I$$

- A subobject:

$$[(k_1, k_2)|k_1 = k_2]^I \xrightarrow{=^I} k^I \times k^I$$

2.2 Terms

Each sort in a signature Σ is equipped with arbitrarily many *variables*. We will write $x{\in}X$ to denote that x is a variable of sort X.

Definition 2.3. A *context* is a list of variables $\vec{x} = (x_1, x_2, ..., x_k)$ such that all the x_i are distinct but several x_i may be of the same sort.

Definition 2.4. Σ-*terms* (in context) are defined inductively as follows:

- Let $x{\in}X$ and $\vec{x}$ be a context containing x. Then the *variable in context* $(\vec{x}|x)$ is a term with *output* X.
- Let

  $$X_1{\times}X_2{\times}...{\times}X_k \xrightarrow{f} Y$$

  be a function symbol of Σ and $(\vec{z_i}|t_i)$ be terms with outputs X_i. Without loss of generality we may assume that there are no repeated variables in the concatenation of the $\vec{z_i}$: if there were we could swap them for unused variables of the same sort. Then the expression

  $$(\vec{z}|ft_1t_2...t_k)$$

  is a term with output Y where $\vec{z}$ is the concatenation of the $\vec{z_i}$ (which we will write $z_1 : z_2 : ... : z_k$).

The interpretation of the variable in context $((x_1, x_2, ..., x_k)|x_i)$ in a category is the ith projection:

$$X_1^I{\times}X_2^I{\times}...{\times}X_k^I \xrightarrow{x_i} X_i^I$$

The interpretation of the term $(\vec{z}|ft_1t_2...t_k)$ is simply the composition:

$$(\vec{z}|ft_1t_2...t_k)^I = f^I{\circ}((\vec{z}_1|t_1), (\vec{z}_2|t_2), ..., (\vec{z}_k|t_k))$$

Example 2.5. The expression $t = ((k_1, k_2, k_3)|(k_1{\times}k_2) + k_3)$ (where $k_i{\in}k$) is a Σ_{ring}-term. Consider the interpretation I in Example 2.2. Then the interpretation of t is the composite:

$$k^I{\times}k^I{\times}k^I \xrightarrow{(\times^I, k_3)} k^I{\times}k^I \xrightarrow{+^I} k^I$$

where the k_3 is the projection from the third factor.

2.3 Formulae

Definition 2.6. An *atomic formula* (in context) is an expression of the form $[\vec{y}|Rt_1t_2...t_k]$ where

$$[(x_1, x_2, ..., x_k)|Rx_1x_2...x_k] \overset{R}{\rightarrowtail} X_1{\times}X_2{\times}...{\times}X_k$$

is a relation symbol, $(\vec{y}_i|t_i)$ are terms with output X_i and again $\vec{y} = \vec{y}_1 : \vec{y}_2 : ... : \vec{y}_k$ with any repeats removed.

Given an interpretation I of a signature Σ we justify our notation by defining the interpretation of an atomic formula $[\vec{y} = (y_1, y_2, ..., y_m)|Rt_1t_2...t_k]$ (where $y_i \in Y_i$) to be the subobject:

$$[\vec{y}|Rt_1t_2...t_k]^I \rightarrowtail Y_1^I \times Y_2^I \times ... \times Y_m^I$$

defined by the pullback:

$$\begin{array}{ccc}
[\vec{y}|Rt_1t_2...t_k]^I & \dashrightarrow & Y_1^I \times Y_2^I \times ... \times Y_m^I \\
\downarrow & & \downarrow (t'_1, t'_2, ..., t'_k) \\
[(x_1, x_2, ..., x_k)|Rx_1x_2...x_k]^I & \xrightarrow{R^I} & X_1^I \times X_2^I \times ... \times X_k^I
\end{array}$$

where t'_i are the arrows defined as the precomposition of $(\vec{y}_i|t_i)^I$ with the appropriate projection. i.e. the projection from $Y_1 \times Y_2 \times ... \times Y_n$ to the domain of $(\vec{y}_i|t_i)^I$.

Example 2.7. Consider the interpretation I of the signature of rings Σ in Example 2.2. Then $((k_1, k_2)|k_1 \times k_2 = k_2 \times k_1)$ is an atomic formula given by the upper arrow in the pullback:

$$\begin{array}{ccc}
[(k_1, k_2)|k_1 \times k_2 = k_2 \times k_1]^I & \dashrightarrow & k^I \times k^I \\
\downarrow & & \downarrow (k_1 \times k_2, k_2 \times k_1) \\
[(k_1, k_2)|k_1 = k_2]^I & \xrightarrow{R^I} & k^I \times k^I
\end{array}$$

Definition 2.8. *Horn formulae* are defined inductively as follows:

- $[\vec{x}|\top]$ is a Horn formula for any context $\vec{x}$
- Each atomic formula is a Horn formula.
- If $[\vec{x}|\phi]$ and $[\vec{x}|\psi]$ are Horn formulae with the same context then $[\vec{x}|\phi] \wedge (\vec{x}|\psi)$ is a Horn formula.

The interpretation of $[(x_1, x_2, ..., x_k)|\top]$, where $x_i \in X_i$, in a category is the trivial subobject of $X_1 \times X_2 \times ... \times X_k$ induced by the identity arrow. A category that has all finite limits is called a *cartesian category*. In particular a cartesian category has all finite products and all pullbacks. In such a category we can interpret $[\vec{x}|\phi] \wedge [\vec{x}|\psi]$ as the pullback of the subobjects $[\vec{x}|\phi]^I$ and $[\vec{x}|\psi]^I$.

2.4 Theories and Models

Definition 2.9. An *algebraic signature* has no relation symbols except for an equality relation symbol

$$[(x_1, x_2)|x_1 = x_2] \rightarrowtail^{=_X} X{\times}X$$

for each sort X.

Example 2.10. The signature of rings Σ_{ring} given in Example 2.2 is an algebraic signature.

We now specify a deduction system using the sequent-calculus. We take as *logical axioms*:

$$\begin{aligned} \phi &\vdash_{\vec{x}} \top \\ (\phi \wedge \psi) &\vdash_{\vec{x}} \phi \\ (\phi \wedge \psi) &\vdash_{\vec{x}} \psi \end{aligned}$$

where we use the notation $\phi \vdash_{\vec{x}} \psi$ to indicate that the context of both ϕ and ψ is $\vec{x}$. We take as rules of inference:

$$\frac{\phi \vdash_{\vec{x}} \psi \qquad \phi \vdash_{\vec{x}} \chi}{\phi \vdash_{\vec{x}} \psi \wedge \phi}$$

$$\frac{}{\psi \vdash_{\vec{x}} \top}$$

$$\frac{}{\psi_1, \psi_2, ..., \psi_m \vdash_{\vec{x}} \psi_i}$$

$$\frac{}{\psi_1, \psi_2, ..., \psi_m, \top \vdash_{\vec{x}} \phi}$$

$$\frac{\phi, (z = y) \vdash_{\vec{x}:z:y} \chi}{\phi \vdash_{\vec{x}:z} \chi\,[z/y]}$$

Definition 2.11. An *algebraic theory* $\mathbb{T}$ over an algebraic signature consists of a set of sequents of the form

$$\top \vdash \Phi_i$$

where Φ_i is an atomic formulae. The members of this set are called the *axioms* of the theory.

Example 2.12. The *theory of rings* $\mathbb{T}_{ring}$ is the theory over Σ_{ring} that, for k_i variables of sort k and $\vec{k} = (k_1, k_2, k_3)$, has the axioms:

$$\begin{aligned} \top \vdash_{\vec{k}} k_1 + (k_2 + k_3) = (k_1 + k_2) + k_3 \\ \top \vdash_{\vec{k}} k_1 + k_2 = k_2 + k_1 \\ \top \vdash_{\vec{k}} k_1 + 0 = k_1 \\ \top \vdash_{\vec{k}} k_1 + (-k_1) = 0 \\ \top \vdash_{\vec{k}} k_1 \times (k_2 \times k_3) = (k_1 \times k_2) \times k_3 \\ \top \vdash_{\vec{k}} k_1 \times 1 = k_1 \\ \top \vdash_{\vec{k}} 1 \times k_1 = k_1 \\ \top \vdash_{\vec{k}} k_1 \times (k_2 + k_3) = k_1 \times k_2 + k_1 \times k_3 \\ \top \vdash_{\vec{k}} (k_1 + k_2) \times k_3 = (k_1 \times k_3) + (k_2 \times k_3) \end{aligned}$$

Thus the theory of rings is an algebraic theory.

Definition 2.13. The *syntactic category* $\mathcal{C}_{\mathbb{T}}$ of an algebraic theory $\mathbb{T}$ has:

- objects are equivalence classes of all Horn formulae $[\vec{x}|\phi]$. Two formulae are equivalent precisely when:

$$[\vec{x}|\phi] \sim [\vec{x}|\psi] \iff (\phi \vdash_{\vec{x}} \psi) \wedge (\psi \vdash_{\vec{x}} \phi)$$

- arrows are equivalence classes of m-tuples of terms:

$$[\vec{y}|\phi] \xrightarrow{((\vec{y}_1|t_1), (\vec{y}_2|t_2), ..., (\vec{y}_3|t_3))} [(x_1, x_2, ..., x_k)|\psi]$$

  where $(\vec{y}_i|t_i)$ has output X_i, $x_i \in X_i$ and $\vec{y} = \vec{y}_1 : \vec{y}_2 : ... : \vec{y}_k$ with any repeats removed. Two m-tuples of terms are equivalent precisely when:

$$\begin{aligned} &((\vec{y}_1|t_1),(\vec{y}_2|t_1), ..., (\vec{y}_k|t_k)) \sim ((\vec{z}_1|s_1), (\vec{z}_2|s_2), ..., (\vec{z}_k|s_k)) \iff \\ &\qquad \phi \vdash_{\vec{x}} ((\vec{y}_1|t_1) = (\vec{z}_1|s_1)) \wedge ((\vec{y}_2|t_1) = (\vec{z}_2|s_2)) \wedge ... \wedge ((\vec{y}_k|t_k) = (\vec{z}_k|s_k)) \wedge \psi \end{aligned}$$

Definition 2.14. A *model* of the theory $\mathbb{T}$ in a cartesian category $\mathcal{D}$ is a finite limit preserving functor $\mathcal{C}_{\mathbb{T}} \to \mathcal{D}$.

Equivalently, a model is an interpretation of the signature, terms and formulae of the theory in a category such that the axioms of the theory hold. Since we are working with algebraic theories the axioms are of the form:

$$\top \vdash_{\vec{x}} t_1 = t_2$$

and are said to hold when the arrows interpreting the terms t_1 and t_2 are the same.

2.5 Generalised Elements

Definition 2.15. An *(generalised) element of R with stage of definition X* is an arrow with codomain R and domain X:

$$r : X \to R$$

and we write $r \in_X R$. If X is clear from the context or we want to reason about a general stage of definition we write simply $r \in R$.

Each arrow $\alpha : Y \to X$ provides (by precomposition) a different way of interpreting r as an element with stage of definition Y. Thus an element that has a terminal object for stage of definition (a *global* element) has unique interpretation as an element with any other stage of definition.

Notation: If $\phi : X \to R^A$ and $a : X \to A$ then define $\phi(a)$ as the composite

$$X \xrightarrow{(a,\phi)} A \times R^A \xrightarrow{ev.} R$$

where $ev.$ is the image of the identity morphism of R^A under the adjunction given by the λ-conversion:

$$\frac{1_{R^A} : R^A \to R^A}{ev. = \overline{1_{R^A}} : A \times R^A \to R}$$

Accordingly we will denote an element $\phi : X \to R^A$ as $(a \mapsto \phi(a))$.

3 The Axiomatic Theory

We now proceed to describe some of the axiomatic theory of Synthetic Differential Geometry.

In 3.1 the notion of a Weil algebra is defined which we will use to formulate the fundamental Kock-Lawvere axiom. The treatment is based on Section F1.1 of Volume 3 of [Johnstone, 2006]. For the applications in Sections 3.4 and 3.5 the property of microlinearity is essential. The discussion of microlinearity in Section 3.3 extends the account given in Chapter 6 of [Shulman, 2006]. Although our definition of microlinearity therefore differs from that in [Lavendhomme, 1996] this reference was still useful for proving some of the Propositions in this subsection.

The first application of the axiomatic theory is an alternative method for solving a partial differential equation and is from [Kock and Reyes, 2006]. The final application again follows [Johnstone, 2006] (in particular Proposition 1.2.15 of Chapter F1.2 of Volume 3) in showing that the set of vector fields over a microlinear object in a smooth topos $(\mathcal{E}, R)$ can be given a R-Lie algebra structure.

3.1 Introduction

To motivate the definitions that follow, let us first try to formulate the ideas mentioned in Section 1 in *Set*. Let R be a $\mathbb{Q}$-algebra which will be thought of as a 'line' and D be an object such that arrows $D \to R$ describe tangents to R. That is we want:

Axiom 3.1. *(Kock-Lawvere) Every arrow $g : D \to R$ is uniquely of the form:*

$$g(d) = a + bd$$

for $d \in D$ and $a, b \in R$. Note that $a = g(0)$.

But, in this context we quickly get a contradiction. Because in the internal logic of *Set* the Law of the Excluded Middle holds we can define $f : D \to R$ such that:

$$f(d) = \begin{cases} 1 & \text{if } d = 0 \\ 0 & \text{otherwise} \end{cases}$$

then by the Kock-Lawvere axiom

$$f(d) = 1 + bd$$

for an unique $B \in R$. But if $0 \neq d \in D$ then $0 = f(d) = 1 + bd$ which upon multiplying by d gives: $0 = d + bd^2 = d$. But this is a contradiction with the Kock-Lawvere axiom since if $D = \{0\}$ then we cannot uniquely determine the 'gradient' b.

3.2 Algebras and Smooth Toposes

Definition 3.2. The *signature of k-algebras* Σ_k (where k is a ring) has:

- One sort A.
- Function symbols described by:

$$1 \underset{0}{\overset{1}{\rightrightarrows}} A \underset{\times}{\overset{+}{\leftleftarrows}} A \times A$$

  and an additional function symbol $k\cdot : A \to A$ (representing scalar multiplication) for every element of the ring k.

- The relation symbol:

$$[(a_1, a_2) | a_1 = a_2] \overset{=_A}{\rightarrowtail} A \times A$$

Definition 3.3. The *theory of k-algebras* $\mathbb{T}_k$ consists of all the equalities between composites of function symbols which express the usual notion of A being a k-algebra.

Definition 3.4. A *k-algebra V* in a topos $\mathcal{E}$ is a product preserving functor

$$V : \mathcal{C}_{\mathbb{T}_k} \to \mathcal{E}$$

where $\mathcal{C}_{\mathbb{T}_k}$ is the syntactic category of the theory of k-algebras. We call $V(A)$ a *k-algebra object*. $Alg_k(\mathcal{E})$ is the full subcategory of $[\mathcal{C}_{\mathbb{T}_k}, \mathcal{E}]$ consisting of functors which preserve products. We will write simply Alg_k for the case when $\mathcal{E} = Set$.

Definition 3.5. A *lined topos over k* is a pair $(\mathcal{E}, R)$ where $\mathcal{E}$ is a topos and R is a k-algebra object in $\mathcal{E}$.

Definition 3.6. The *(internal) Weil algebra over B* for a k-algebra object B is a functor $\mathcal{W} : \mathcal{C}_{\mathbb{T}_k} \to \mathcal{E}$ such that:

- $\mathcal{W}(A) = B^n$.
- $\mu := \mathcal{W}(\times) : B^{2n} \to B^n$ is bilinear in B and makes all elements of B^n of the form $(0, a_2, a_3, ..., a_n)$ nilpotent.
- $\mathcal{W}(1) = (1, 0, 0, ..., 0) : 1 \to B^n$.

We say that $W = \mathcal{W}(A)$ is a *Weil algebra object* and n is the *dimension* of the Weil algebra.

Example 3.7. Let $n = 2$ and $\mu = \mu_\epsilon$ where $\mu(a, b, c, d) = (ac, bc + ad)$. This is 'ring of dual numbers multiplication' reflects multiplication in the ring $\mathbb{R}[X]/(X^2)$.

Definition 3.8. An *internal R-algebra* with R-algebra object V in a lined topos $(\mathcal{E}, R)$ consists of the following arrows:

$$\begin{array}{ccccc} 1 & \overset{1}{\underset{0}{\rightrightarrows}} & V & \overset{+}{\underset{\times}{\leftleftarrows}} & V{\times}V \\ & & \Big\uparrow {\scriptstyle \_\cdot\_} & & \\ & & R{\times}V & & \end{array} \tag{3.1}$$

which satisfy all the equalities that express the usual notion of being an algebra over R with scalar multiplication $\_ \cdot \_$.

Definition 3.9. An *R-algebra homomorphism* between two internal R-algebras V and W in a lined topos $(\mathcal{E}, R)$ is an arrow

$$\alpha : V \rightarrow W$$

that commutes with the arrows in Diagram 3.1. That is for example:

$$\begin{array}{ccc} 1 & \xrightarrow{1_V} & V \\ & {\scriptstyle 1_W}\searrow & \Big\downarrow \alpha \\ & & W \end{array}$$

and

$$\begin{array}{ccc} V{\times}V & \xrightarrow{+_V} & V \\ \Big\downarrow (\alpha,\alpha) & & \Big\downarrow \alpha \\ W{\times}W & \xrightarrow{+_W} & W \end{array}$$

commute. We write $RAlg$ for the category with objects all internal R-algebras and arrows all R-algebra homomorphisms.

Definition 3.10. If W is a Weil algebra object then define

$$Spec_R(W) := RAlg(W, R)$$

We will also write $Spec_R(W)$ as D_W when we are working with fixed R.

The following Proposition is Exercise 4.2 in Part II of [Kock, 2006].

Proposition 3.11. *$RAlg(B, R)$ is an object of $\mathcal{E}$ for all $B \in RAlg$.*

Proof. Consider the two arrows:

$$B{\times}B{\times}R^B \xrightarrow{(+,1)} B{\times}R^B \xrightarrow{ev.} R$$

$$B{\times}B{\times}R^B \xrightarrow{(1,1,\Delta)} B{\times}B{\times}R^B{\times}R^B \xrightarrow{\cong} B{\times}R^B{\times}B{\times}R^B \xrightarrow{(ev.,ev.)} R$$

They correspond (after a lambda conversion) to arrows $R^B{\to}R^{B\times B}$. Take the equaliser of these two arrows. In terms of generalised elements the equaliser object will have elements f that satisfy

$$f(b_1 + b_2) = f(b_1) + f(b_2)$$

for b_i elements of B. Repeat this process for multiplication and scalar multiplication in place of addition to obtain a subobject of R^B whose elements are R-algebra homorphisms from B to R. □

Corollary 3.12. *$Spec_R(W)$ is an object of $\mathcal{E}$.*

Example 3.13. Let W be the Weil algebra object with dimension $n = 2$ and μ_ϵ as above. Then $\phi{\in}Spec_R(W)$ means that $\phi(a, b) = a + b\phi(0, 1)$ and:

$$\begin{aligned}(a + b\phi(0, 1))(c + d\phi(0, 1)) &= (\phi(a, b))(\phi(c, d)) = \phi(\mu(a, b, c, d))\\ &= \phi(ac, ad + bc) = ac + (ad + bc)\phi(0, 1)\end{aligned}$$

and so $\phi(0, 1)^2 = 0$. Therefore there is an isomorphism between $Spec_R(W)$ and the object $\{r{\in}R|r^2 = 0\}$ which is defined in the equaliser:

$$\{r{\in}R|r^2 = 0\} \longrightarrow R \underset{0}{\overset{(-)^2}{\rightrightarrows}} R$$

Now consider a general Weil algebra object W. Let ϕ be a general element of $Spec_R(W)$ and let

$$\phi_i = \phi(0, 0, ..., 0, 1, 0, ..., 0)$$

where the 1 is in the ith place. Then the ϕ_i satisfy certain polynomial equations $p_j(\phi_1, \phi_2, ..., \phi_n) = 0$ over R. (In particular, by the definition of Weil algebra, each of the ϕ_i are nilpotent.)

The set of polynomial equations $p_j(\phi_1, \phi_2, ..., \phi_n) = 0$ that are satisfied by *all* $\phi{\in}Spec_R(W)$ we will call the *presentation* of W. Note that all Weil algebra objects W are *finitely presented* in that there is a finite set X of polynomials over R such that if all the polynomials in X are equal to zero then all of the equations in the presentation hold.

Furthermore, if W has the smallest presentation containing:

$$\{(\phi_1^{c_1} = 0), (\phi_2^{c_2} = 0), ..., (\phi_k^{c_k} = 0)\}{\cup}\bigcup_j\{(p_i(\phi_1, \phi_2, ..., \phi_k) = 0)\}$$

then we have that $Spec_R(W)$ is isomorphic to:

$$\{(r_1, r_2, ..., r_n) \in R^n | (r_1^{c_1} = 0) \wedge (r_2^{c_2} = 0) \wedge ... \wedge (r_k^{c_k} = 0) \wedge \bigwedge_{i=1}^{n} (p_i(r_1, r_2, ..., r_k) = 0)\}$$

the object which is defined by the appropriate equaliser (c.f. Example 3.13). We will often identify these two isomorphic objects.

Notation: Let W be a Weil algebra object over k in (Set, k). Then for a k-algebra object B we write

$$B \otimes_k W \tag{3.2}$$

for the Weil algebra object over B that has the same presentation as W.

Definition 3.14. A *smooth topos over k* is a lined topos $(\mathcal{E}, R)$ such that for all Weil R-algebra objects W:

1. $(-)^{Spec_R(W)}$ has a right adjoint.
2. (Kock-Lawvere) The map

   $$\alpha_W : R \otimes_k W \to R^{Spec_R(W)} \tag{3.3}$$

   defined by

   $$w \mapsto (\phi \mapsto \phi(w))$$

   is an isomorphism of R-algebras.

Example 3.15. Let W be defined by the pair $(R(0,1)^2, \mu_\epsilon)$. Then the isomorphism given by the Kock-Lawvere condition is:

$$(a, b) \mapsto (\phi \mapsto \phi(a, b) = a + b\phi(0, 1))$$

Therefore all maps $\phi \in Spec_R(W)$ are uniquely of the form $a + b\phi(0, 1)$ where $\phi(0, 1)$ is a generalised element of R with nilsquare. Thus condition 2 is a generalisation of the original Kock-Lawvere axiom we considered in the introduction.

The first condition in the definition of smooth topos expresses the notion that the object $Spec_R(W)$ is 'infinitesimal' or 'tiny'. This essay will not explore the consequences of this definition and although the model constructed in Section 4 does satisfy it we shall only give an outline of its verification. For more details see F1.4 of Volume 3 of [Johnstone, 2006] and Chapters I.19 and III.8 of [Kock, 2006].

3.3 Microlinearity

Let $P : \mathcal{J} \to \mathcal{C}$ be a diagram. Let τ_1 be a cocone under P in $\mathcal{C}$ with base L. Then we define:

Definition 3.16. L is a *B-colimit* under P if any cocone τ_2 over P with summit B factors uniquely through τ_1.

In the following $(\mathcal{E}, R)$ is a smooth topos.

Definition 3.17. An *infinitesimal diagram* in $\mathcal{E}$ is a diagram $P : \mathcal{J} \to \mathcal{C}$ in which all the objects in the image of P are of the form D_W (that is the spectrum of some Weil algebra object W). We also allow the terminal object of $\mathcal{E}$ to be an object in the image of the diagram P.

Definition 3.18. $M \in \mathcal{E}$ is *microlinear* if all R-colimits of infinitesimal diagrams are M-colimits.

Notation: We write $D(p)$ for the object

$$\{(r_1, r_2, ..., r_p) \in R^p | r_i \times r_j = 0 (\forall 1 \leq i, j \leq p))\}$$

in $\mathcal{E}$ and we write $D = D(1) = \{r \in R | r^2 = 0\}$.

Proposition 3.19. *In $\mathcal{E}$ the diagram:*

$$\begin{array}{ccc} 1 & \xrightarrow{0} & D(q) \\ \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \iota_2} \\ D(p) & \xrightarrow{\iota_1} & D(p+q) \end{array}$$

is an R-pushout, where

$$\iota_1(a_1, a_2, ..., a_p) = (a_1, a_2, ..., a_p, 0, 0, ..., 0)$$

and

$$\iota_2(a'_1, a'_2, ..., a'_q) = (0, 0, ..., 0, a'_1, a'_2, ..., a'_q)$$

Proof. Consider f_1, f_2 such that:

$$\begin{array}{ccc} 1 & \xrightarrow{0} & D(q) \\ \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle f_2} \\ D(p) & \xrightarrow{f_1} & R \end{array} \tag{3.4}$$

commutes. By the Kock-Lawvere axiom we can identify:

$$f_1 = a_0 + a_1 d_1 + ... + a_p d_p$$

and

$$f_2 = a'_0 + a'_1 d'_1 + ... + a'_p d'_p$$

By commutativity of Diagram 3.4 we get that $a_0 = a'_0$. Next we see that f_1 and f_2 factor via $h : D(p+q) \to R$ defined by:

$$h = a_0 + a_1 d_1 + ... + a_p d_p + a'_1 d_{p+1} + ... a'_q d_{p+q}$$

(i.e. $h \circ \iota_1 = f_1$ and $h \circ \iota_2 = f_2$).

For uniqueness of the factorisation, suppose that

$$k = b_0 + b_1 d_1 + ... + b_{p+q} d_{p+q}$$

satsifies $k \circ \iota_1 = f_1$ and $k \circ \iota_2 = f_2$. Now setting $d_i = 0$ for all i we find $b_0 = a_0$. Subsequently setting $d_i = 0$ for all i except for d_j gives:

$$b_j = a_j \text{ when } j \leq p$$

$$b_j = a'_{j-p} \text{ when } j > p$$

that is $k = h$. □

Definition 3.20. A *double-coequaliser* is the colimit of a diagram of the form:

$$A \overset{a,b,c}{\Rrightarrow} B$$

Proposition 3.21. *The diagram in $\mathcal{E}$:*

$$D \overset{\iota_1, \iota_2, 0}{\Rrightarrow} D \times D \xrightarrow{\times} D$$

is an R-double-coequaliser.

Proof. Write $f : D \times D \to R$ as

$$f = a_0 + a_1 d_1 + a_2 d_2 + a_3 d_1 d_2$$

and suppose that

$$D \overset{\iota_1, \iota_2, 0}{\Rrightarrow} D \times D \xrightarrow{f} R$$

commutes. Then

$$f(0,0) = a_0 = f(d,0) = f(0,d)$$

so taking $d_1 = 0$ we find $a_2 = 0$ and taking $d_2 = 0$ we find $a_1 = 0$. Therefore

$$f = a_0 + a_3 d_1 d_2$$

and thus

$$f = h \circ \times$$

where $h = a_0 + a_3 d$. Therefore we have the required factorisation. Suppose that $k = a + bd'$ and that $k \circ \times = f$ then $a = a_0$ and $a_3 = b$ so we have required uniqueness of factorisation. □

We now prove a proposition that will be useful when we consider the Lie algebra of vector fields in Section 3.5.

Proposition 3.22. *If M is microlinear then M^M is microlinear also.*

Proof. Let $P : \mathcal{J} \to \mathcal{E}$ be a diagram with vertices P_i. Let $\tau : P \Rightarrow L$ be an M-colimit under P. Let $\sigma : P \Rightarrow M^M$ be another cocone over P that has base M^M. For each $m \in M$ we have that

$$\phi_m \circ \sigma$$

(where ϕ_m is evaluation at m) determines a cocone under P that has base M. Since $\tau : P \Rightarrow L$ is an M-colimit there exist unique arrows $g_m : L \to M$ inducing factorisations of $\phi_m \circ \sigma : P \Rightarrow M$ through $\tau : P \Rightarrow L$. Combining these gives a factorisation of $(\sigma)_i$ through $((\tau)_i, L)$.

If there were two distinct arrows $f_1, f_2 : L \to M^M$ that induced factorisations of $\sigma : P \Rightarrow M^M$ through $\tau : P \Rightarrow L$ then we could find $l \in L$ such that $f_1(l) \neq f_2(l)$. Then let $\phi_l : M^M \to M$ denote the evaluation at l:

$$\phi_l(m \mapsto \psi(m)) = \psi(l)$$

We see that $\phi_l \circ f_1$ and $\phi_l \circ f_2$ are two distinct factorisations of $\phi_l \circ (\sigma)$ (a cocone with base M) through $\tau : P \Rightarrow L$. This contradiction shows the L is an M^M-colimit also. □

3.4 Derivatives and Differential Equations

In the following $(\mathcal{E}, R)$ will be a smooth topos over k.

Definition 3.23. An *infinitesimally linear* object M in $(\mathcal{E}, R)$ is one for which all n-tuples of maps

$$t_i : D \to M$$

such that $t_1(0) = t_2(0) = ... = t_n(0)$ there exists an unique $l : D(n) \to M$ such that

$$l \circ \iota_i = t_i$$

for all i, where ι_i is the ith injection $D : \to D(n)$.

Corollary 3.24. *If M is microlinear in $(\mathcal{E}, R)$ then M is infinitesimally linear.*

Proof. This follows from Proposition 3.19. □

Definition 3.25. A *Euclidean object* V in a smooth topos $(\mathcal{E}, R)$ is one for which every arrow

$$D \to V$$

is of the form

$$d \mapsto a + bd$$

Definition 3.26. A *tangent vector* in M is an arrow $D \to M$ in $(\mathcal{E}, R)$.

Definition 3.27. The *tangent bundle* over M is the object M^D in $(\mathcal{E}, R)$.

Definition 3.28. A *vector field* over M is an arrow $\hat{X} : M \to M^D$ such that

$$\hat{X}(m) = (d \mapsto m + \xi(m)d)$$

for some $\xi : M \to M$. We call ξ the *principal part* of $\hat{X}$.

Note that any element of M^D is of the form $d \mapsto a + bd$ by the fact that M is Euclidean.

Now we make the λ-conversion:

$$\frac{\hat{X} : M \to M^D}{X : D \times M \to M}$$

Therefore a vector field on M is essentially the same as an infinitesimal action $X : D \times M \to M$ such that $X(d, m) = m + \xi(m)d$.

We then make a further λ-conversion:

$$\frac{X : D \times M \to M}{\check{X} : D \to M^M}$$

to see that a vector field on M is essentially the same as an infinitesimal transformation $\check{X} : D \to M^M$ such that $\check{X}(d) = (m \mapsto m + \xi(m)d)$.

Proposition 3.29. *Let M be microlinear and X be a vector field on M. Then for all $(d_1, d_2) \in D(2)$:*

$$f(d_1, d_2) := X(d_2, X(d_1, m)) = X(d_1 + d_2, m) =: g(d_1, d_2)$$

Proof. We will show that $f \circ \iota_1 = g \circ \iota_1$ and $f \circ \iota_2 = g \circ \iota_2$ and conclude that $f = g$ by Corollary 3.24.

$$f(0, d_2) = X(d_2, X(0, m)) = X(d_2, m) = g(0, d_2)$$

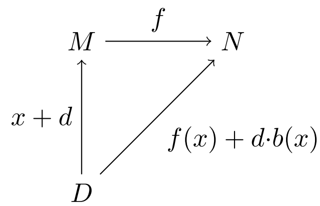
$$f(d_1, 0) = X(0, X(d_1, m)) = X(d_1, m) = g(d_1, 0)$$

□

Corollary 3.30. *Let M be microlinear and X a vector field on M. Then:*

$$X(d, m)^{-1} = X(-d, m)$$

If M, N are two Euclidean objects then we can define the derivative f' of a map $f : M \to N$ at x as the unique $b(x) \in N$ that makes the following diagram commute:

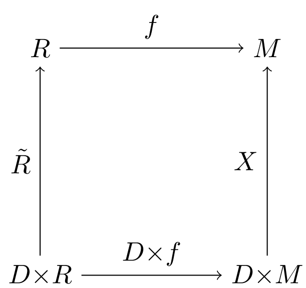


Now we describe how a vector field describes a differential equation. Let $\tilde{R} : D \times R \to R$ be the *universal action* on R described by

$$\tilde{R}(d, r) = r + d$$

Intuitively we think of a solution of the differential equation defined by a vector field on an Euclidean manifold as the locus traced out when we place an object at some point on the manifold and it moves in the direction defined by the vector field. Thus:

Definition 3.31. A *solution to the differential equation described by* X is an arrow $f : R \to M$ such that



commutes. The *initial value* of the solution is $f(0)$

If ξ is the principal part of X then we have that $f(r + d) = f(r) + d \cdot \xi(f(r))$. Thus the differential equation described by X is:

$$\frac{df}{dr} = \xi(f)$$

Now let X, Y be vector fields over Euclidean and microlinear objects M, N respectively and having principal parts ξ, η respectively. Let $f_{m_0} : R \to M$ be a solution of X with initial value m_0 and $g_{n_0} : R \to N$ a solution of Y with initial value n_0. Assume that $f_{(-)}(r)$ is invertible for all $r \in R$: that is if we know the value of the solution at r then we can uniquely determine its initial value.

We will define an *exponential action* on the exponential N^M and interpret the resulting vector field as a differential equation. The exponential action Y^X on N^M we will define as:

$$Y^X(d, \beta) = Y(d, -)\circ\beta\circ X(-d, -)$$

Firstly, we see that

$$h(r) = g_{(-)}(r)\circ\beta\circ f_{(-)}(r)^{-1}$$

is a solution of Y^X with initial value $\beta = h(0)$:

$$\begin{aligned} Y^X\left(d, g_{(-)}(r)\circ\beta\circ f_{(-)}(r)^{-1}\right) &= Y(d, -)\circ g_{(-)}(r)\circ\beta\circ f_{(-)}(r)^{-1}\circ X(d, -)^{-1} \\ &= Y(d, g_{(-)}(r))\circ\beta\circ X(d, f_{(-)}(r))^{-1} \\ &= g_{(-)}(r + d)\circ\beta\circ f_{(-)}(r + d)^{-1} \\ &= g_{(-)}\left(\tilde{R}(d, r)\right)\circ\beta\circ\left(f_{(-)}(\tilde{R}(d, r))\right)^{-1} \end{aligned}$$

So we can find the solution to the differential equation described by the vector field Y^X. But what is the actual differential equation? We need to find the principal part of Y^X:

$$\begin{aligned} Y^X(d, \beta)(m) &= Y(d, -)\circ\beta\circ X(-d, m) \\ &= Y(d, -)\circ\beta(m - d\cdot\xi(m)) \\ &= \beta(m - d\cdot\xi(m)) + d\cdot\eta(m - d\cdot\xi(m)) \\ &= \beta(m) - d\cdot\eta(m)\cdot\beta'(m) + d\cdot\eta(m) \end{aligned}$$

Thus the principal part of Y^X is:

$$\eta - \xi\cdot\beta'$$

and so the differential equation described by the vector field Y^X is:

$$\frac{\partial h}{\partial t} = \eta(h) - \xi(m)\frac{\partial h}{\partial m} \tag{3.5}$$

In summary: if we can find a solution f of

$$\frac{dF}{dr} = \xi(F)$$

and a solution g of

$$\frac{dG}{dr} = \eta(G)$$

then a solution to the differential equation 3.5 with initial value β is given by $h(r) = g_{(-)}(r)\circ\beta\circ f_{(-)}(r)^{-1}$.

3.5 The Lie Algebra of Vector Fields

In the following $(\mathcal{E}, R)$ will be a smooth topos over k.

Let X and Y be vector fields over a microlinear object M. Then we have

$$\check{X}, \check{Y} : D \to M^M$$

Now define the sum $\check{X} + \check{Y} : D \to M^M$ to be the composite:

$$D \xrightarrow{\quad \Delta \quad} D(2) \xrightarrow{\quad \langle \check{X}, \check{Y} \rangle \quad} M^M$$

where Δ is the diagonal and $\langle \check{X}, \check{Y} \rangle$ is the arrow $D(2) \to M^M$ whose restrictions to the axes are $\check{X}$ and $\check{Y}$ respectively. The existence and uniqueness of $\langle \check{X}, \check{Y} \rangle$ is given by Proposition 3.19. Since X and Y are vector fields we see that $\check{X}(0) = \check{Y}(0) = id_M$ and so:

$$(\check{X} + \check{Y})(d) = \langle \check{X}, \check{Y} \rangle (d, d) = \check{X}(d)\check{Y}(d) = \check{Y}(d)\check{X}(d)$$

Define multiplication by $r \in R$ as

$$(r \cdot X)(d, m) = X(rd, m)$$

It is straightforward to see that the above definitions of addition and scalar multiplication define an R-module structure on the set of vector fields over M.

Definition 3.32. An object M of $\mathcal{E}$ *has Property W* if for any $\tau : D \times D \to M$ with

$$\tau(d, 0) = \tau(0, d) = \tau(0, 0)$$

for all $d \in D$ then there exists an unique $t : D \to M$ with

$$\tau(d_1, d_2) = t(d_1 \cdot d_2)$$

Corollary 3.33. *If M is microlinear then M has Property W.*

Proof. This is a consequence of Proposition 3.21. □

Recall that if M is microlinear then M^M is also microlinear (Proposition 3.22) and consider the map $\tau : D \times D \to M^M$ defined by

$$\tau(d_1, d_2) = \check{X}(-d_1) \circ \check{Y}(-d_2) \circ \check{X}(d_1) \circ \check{Y}(d_2)$$

Then τ satisfies Property W and so there exists an unique (vector field)

$$[\check{X}, \check{Y}] : D \to M^M$$

such that $[\check{X}, \check{Y}](d_1 \cdot d_2) = \tau(d_1, d_2)$. In fact we have:

Proposition 3.34. *With $[-,-]$ as a Lie bracket the set of vector fields over M is an R-Lie algebra. That is:*

(i) $[-,-]$ *is bilinear.*

(ii) $[X,Y]+[Y,X]=0$.

(iii) $[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0$.

Proof. We have already defined the addition and scalar multiplication that gives the R-module structure. If we prove (i) then we only need to check that $[-,-]$ is bilinear in the second variable. Finally we will not verify (iii) - the details can be found in section 3.2.2 of [Lavendhomme, 1996] or Proposition 1.2.15 of Chapter F1.2 of Volume 3 of [Johnstone, 2006].

Antisymmetry:

$$\begin{aligned}
[\check{X},\check{Y}](d_1{\cdot}d_2) &= [\check{X},\check{Y}](-d_1{\cdot}d_2)^{-1} \\
&= \left(\check{X}(d_1)\circ\check{Y}(-d_2)\circ\check{X}(-d_1)\circ\check{Y}(d_2)\right)^{-1} \\
&= \check{Y}(-d_2)\circ\check{X}(d_1)\circ\check{Y}(d_2)\circ\check{X}(-d_1) \\
&= [\check{Y},\check{X}](d_1{\cdot}d_2) \\
&= \left(-[\check{Y},\check{X}]\right)(d_1{\cdot}d_2)
\end{aligned}$$

Bilinearity in second variable:

$$[\check{X},\lambda\check{Y}](d_1{\cdot}d_2) = [\check{X},\check{Y}](d_1{\cdot}\lambda d_2) = \lambda[\check{X},\check{Y}](d_1{\cdot}d_2)$$

In the following let $\{x,y\}$ denote the group-theoretic commutator $x^{-1}y^{-1}xy$.

$$\begin{aligned}
&[\check{X},\check{Y}+\check{Z}](d_1{\cdot}d_2) = \check{X}(-d_1)\circ\check{Y}(-d_2)\circ\check{Z}(-d_2)\circ\check{X}(d_1)\circ\check{Y}(d_2)\circ\check{Z}(d_2) \\
&= \left[\check{X}(-d_1)\circ\check{Y}(-d_2)\circ\check{X}(d_1)\circ\check{Y}(d_2)\right]\circ\check{Y}(-d_2)\circ\check{X}(-d_1)\circ\check{Z}(-d_2)\circ\check{X}(d_1)\circ\check{Z}(d_2)\circ\check{Y}(d_2) \\
&= \{\check{X}(d_1),\check{Y}(d_2)\}\circ\check{Y}(-d_2)\circ\{\check{X}(d_1),\check{Z}(d_2)\}\circ\check{Y}(d_2)
\end{aligned}$$

So we are done if we can show that $\check{Y}(d_2)$ commutes with $\{\check{X}(d_1),\check{Z}(d_2)\}$. To show this we use the fact that $[\check{Y},[\check{X},\check{Z}]]$ is a vector field:

$$id_M = [\check{Y},[\check{X},\check{Z}]](0) = [\check{Y},[\check{X},\check{Z}]](d_2{\cdot}d_1{\cdot}d_2)$$

and we are done. □

Thus the readily visualizable concept that is the commutator of two infinitesimal transformations defines a Lie bracket.

4 C^∞-algebras

This section introduces the theory of C^∞-algebras which will be used to construct a model of a smooth topos in Section 4.

We first describe a relationship between certain C^∞-algebras and certain $\mathbb{R}$-algebras using Hadamard's Lemma (Lemma 4.7). This is done in the same way as in Chapter III.5 of [Kock, 2006]. We will use this connection to prove the Kock-Lawvere axiom for our chosen model in Section 4.

We then introduce the idea of a germ-determined C^∞-algebra. When we construct our model in Section 4 we will want some kind of analogue of an open covering of a smooth manifold. This will done with a suitable coverage (called the Debuc coverage). The restriction to (finitely generated) germ-determined C^∞-algebras is then required to ensure that the Debuc coverage is subcanonical: an important property that will be used in verifying both the Kock-Lawvere axiom and that Weil spectra are tiny in our model. Chapter III.6 of [Kock, 2006] and F1.3 of Volume III of [Johnstone, 2006] are used extensively in this subsection (3.2).

The final subsection will outline the argument that the ring of smooth functions $M \to \mathbb{R}$ for a smooth manifold M is finitely presented. Amongst other things, this is important for showing that the category of smooth manifolds embeds into our model.

4.1 C^∞-algebras and Hadamard's Lemma

Definition 4.1. The *signature of C^∞-algebras* Σ_∞ has:

- One sort A.
- A function symbol
$$A^n \to A$$
for every smooth function $\mathbb{R}^n \to \mathbb{R}$.
- The relation symbol:
$$[(a_1, a_2) | a_1 = a_2] \rightarrowtail^{=_A} A \times A$$

In particular all function/relation symbols in the signature of $\mathbb{R}$-algebras are function/relation symbols in the signature of C^∞-algebras.

Definition 4.2. The *theory of C^∞-algebras* $\mathbb{T}_\infty$ has for axioms all the equalities between composites of function symbols of the form $k^n \to k$ which hold for the corresponding smooth functions $\mathbb{R}^n \to \mathbb{R}$.

In particular all the equalities that hold in the theory of $\mathbb{R}$-algebras also hold in the theory of C^∞-algebras.

Definition 4.3. An *C^∞-algebra V* in a topos $\mathcal{E}$ is a product preserving functor

$$V : \mathcal{C}_{\mathbb{T}_\infty} \to \mathcal{E}$$

where $\mathcal{C}_{\mathbb{T}_\infty}$ is the syntactic category of the theory of C^∞-algebras. We call $V(A)$ a *C^∞-algebra object*. $Alg_\infty(\mathcal{E})$ is the full subcategory of $[\mathcal{C}_{\mathbb{T}_\infty}, \mathcal{E}]$ consisting of functors which preserve products. We will write simply Alg_∞ for the case when $\mathcal{E} = Set$.

We will now introduce the category of smooth manifolds which will embed into our model in Section 4. For the purposes of this essay a smooth manifold will be also Hausdorff and paracompact:

Definition 4.4. A *paracompact* smooth manifold is one for which every open cover of a set $(V_i)_i$ can be refined to a cover $(U_j)_j$ such that:

- Each U_j is contained in one of the V_i.
- For every point $p \in M$ we have that there exists an open neighbourhood W of p such that the set of U_j that meet W is finite.

We call such a $(U_j)_j$ a *locally finite refinement* of $(V_i)_i$

Definition 4.5. The category Man has objects as (Hausdorff, paracompact and) smooth manifolds and arrows as smooth functions between them.

Example 4.6. Define

$$C^\infty(M) := Man(M, \mathbb{R})$$

where $M \in Man$. Then:

- $C^\infty(M)$ is a C^∞-algebra.
- $C^\infty(\mathbb{R})$ is the free C^∞-algebra on one generator.

and $C^\infty(-)$ is a contravariant hom-functor $Man \to Alg_\infty^{op}$.

Lemma 4.7. *(Hadamard's Lemma) Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be smooth. Then there exist smooth $\psi_i : \mathbb{R} \to \mathbb{R}$ such that:*

$$\phi(\vec{x}) = \phi(\vec{a}) + \sum_{i=1}^{n} (x_i - a_i)\psi_i(\vec{x}) \tag{4.1}$$

where $\vec{x} = (x_1, x_2, ..., x_n)$ and $\vec{a} = (a_1, a_2, ..., a_n)$

Proof. See Theorem 2.8 of [Nestruev, 2003] □

Iterating once, we obtain the result that there exist smooth $\psi_i, \nu_j : \mathbb{R}{\to}\mathbb{R}$ such that:

$$\begin{aligned}
\phi(\vec{x}) &= \phi(\vec{a}) + \sum_{i=1}^{n}(x_i - a_i)\left(\psi_i(\vec{a}) + \sum_{j=1}^{n}(x_j - a_j)\nu_j(\vec{x})\right) \\
&= \phi(\vec{a}) + \sum_{i=1}^{n}(x_i - a_i)\psi_i(\vec{a}) + \sum_{i,j<n+1}(x_i - a_i)(x_j - a_j)\nu_j(\vec{x})
\end{aligned}$$

Iterating further and setting $\vec{a} = \vec{x} + \vec{y}$ we obtain the result that there exist smooth $\psi_{\vec{\beta}} : \mathbb{R}^n{\to}\mathbb{R}$ and there exist unique $\phi_{\vec{\alpha}} : \mathbb{R}^{2n}{\to}\mathbb{R}$ such that:

$$\phi(\vec{x} + \vec{y}) = \sum_{|\alpha|<l+1} \phi_{\vec{\alpha}}(\vec{x}){\cdot}\vec{y}^{\vec{\alpha}} + \sum_{|\beta|=l+1} \psi_{\beta}(\vec{x}, \vec{y}){\cdot}\vec{y}^{\vec{\beta}} \tag{4.2}$$

where $k, \alpha_i, \beta_i{\in}\mathbb{N}$, $\vec{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_n)$ and $\vec{\beta} = (\beta_1, \beta_2, ..., \beta_n)$ are indicies varying over the sum with $|\vec{\alpha}| = \sum_i \alpha_i$ and:

$$\vec{y}^{\vec{\alpha}} = y_1^{\alpha_1}{\cdot}y_2^{\alpha_2}{\cdot}...{\cdot}y_n^{\alpha_n}$$

We now use Equation 4.2 to relate coproducts in $Alg_{\mathbb{R}}$ and Alg_{∞}. Note that the coproduct in $Alg_{\mathbb{R}}$ of C and D is the tensor product $C{\otimes_{\mathbb{R}}}D$ and when D is a Weil algebra over $\mathbb{R}$ the use of the symbol $\otimes$ for a tensor product coincides with its use in Equation 3.2.

Let W be a Weil algebra object over $\mathbb{R}$ in $(Set, \mathbb{R})$ and $B{\in}Alg_{\infty}$. Consider $\bar{B}{\otimes_{\mathbb{R}}}W$ where $\bar{B}$ is B is taken with its $\mathbb{R}$-algebra structure only. Then each element of $\bar{B}{\otimes_{\mathbb{R}}}W$ is of the form

$$\iota_{\bar{B}}(b) + d$$

for d nilpotent and $\iota_{\bar{B}} : \bar{B}{\hookrightarrow}\bar{B}{\otimes_{\mathbb{R}}}W$ the inclusion given in the coproduct.

Then, using Equation 4.2, we may define a C^{∞}-algebra structure on $\bar{B}{\otimes_{\mathbb{R}}}W$.

First see that for each $\phi : \mathbb{R}{\to}\mathbb{R}$ and

$$\iota_B(\vec{b}) + \vec{d} = (\iota_B(b_1) + d_1, \iota_B(b_2) + d_2, ..., \iota_B(b_n) + d_n)$$

then

$$\begin{aligned}
\phi(\iota_B(\vec{b}) + \vec{d}) &= \sum_{|\alpha|<l+1} \phi_{\vec{\alpha}}(\iota_B(\vec{b})){\cdot}\vec{d}^{\vec{\alpha}} + \sum_{|\beta|=l+1} \psi_{\beta}(\iota_B(\vec{b}), \vec{d}){\cdot}\vec{d}^{\vec{\beta}} \\
&= \sum_{|\alpha|<l+1} \phi_{\vec{\alpha}}(\iota_B(\vec{b})){\cdot}\vec{d}^{\vec{\alpha}}
\end{aligned}$$

where l is the least natural number such that $d_i^l = 0$ for all i, and the ϕ_{α_i} are unique. Then we can *define* a canonical C^∞-algebra structure $(B{\otimes}_{\mathbb{R}}W)_\infty$ on $B{\otimes}_{\mathbb{R}}W$ by letting

$$\phi(\iota_B(\vec{b}) + \vec{d}) = \sum_{|\alpha|<l+1} \phi_{\vec{\alpha}}(\iota_B(\vec{b}))\cdot\vec{d}^{\vec{\alpha}} = \sum_{|\alpha|<l+1} \iota_B(\phi_{\vec{\alpha}}(\vec{b}))\cdot\vec{d}^{\vec{\alpha}} \tag{4.3}$$

By construction the above equation defines the unique C^∞-algebra structure on $B{\otimes}_{\mathbb{R}}W$ that extends the $\mathbb{R}$-algebra structure *and* makes ι_B a C^∞-algebra homomorphism.

Proposition 4.8. *Let W be a Weil algebra over $\mathbb{R}$ in $(Set, \mathbb{R})$. The C^∞-algebra structure $(\bar{B}{\otimes}_{\mathbb{R}}W)_\infty$ defined on $\bar{B}{\otimes}_{\mathbb{R}}W$ by Equation 4.3 is the coproduct:*

$$B +_\infty W_\infty$$

in Alg_∞ where W_∞ is the canonical C^∞-algebra structure which extends the $\mathbb{R}$-algebra structure on W.

Proof. By construction of the C^∞-algebra structure on $\bar{B}{\otimes}_{\mathbb{R}}W$ the injection $\iota_{\bar{B}} : \bar{B}{\to}\bar{B}{\otimes}_{\mathbb{R}}W$ is a C^∞-algebra homomorphism $B{\to}(\bar{B}{\otimes}_{\mathbb{R}}W)_\infty$. Now consider an $\mathbb{R}$-algebra of the form

$$\bar{A}{\otimes}_{\mathbb{R}}W$$

where A is a C^∞-algebra. Then since the $\mathbb{R}$-algebra structure on $A{\otimes}_{\mathbb{R}}W$ completely determines the C^∞-algebra structure on $(\bar{A}{\otimes}_{\mathbb{R}}W)_\infty$ we have that $\mathbb{R}$-algebra homomorphisms out of $\bar{A}{\otimes}_{\mathbb{R}}W$ are in fact C^∞-algebra homomorphisms out of $(\bar{A}{\otimes}_{\mathbb{R}}W)_\infty$. Thus $\iota_W : W{\to}\bar{B}{\otimes}_{\mathbb{R}}W$ is also a C^∞-algebra homomorphism.

Now let $\phi : W{\to}X$ and $\psi : \bar{B}{\to}X$ be C^∞-algebra homomorphisms. By forgetting the extra structure we see that there is a unique $\mathbb{R}$-algebra homomorphism $\sigma : \bar{B}{\otimes}_{\mathbb{R}}W{\to}X$ that has $\sigma{\circ}\iota_W = \phi$ and $\sigma{\circ}\iota_B = \psi$. But since σ is also a C^∞-algebra homomorphism (it has domain $\bar{B}{\otimes}_{\mathbb{R}}W$) this proves the proposition. □

Using the notation above (and that introduced in Equation 3.2) it is clear that

$$(\bar{B}{\otimes}_{\mathbb{R}}W)_\infty{\cong}B{\otimes}_{\mathbb{R}}W$$

where the left hand side is the canonical C^∞-algebra structure on an $\mathbb{R}$-algebra and the right hand side is an (internal) Weil algebra over a C^∞-algebra B. Thus we obtain by the Proposition above:

$$B +_\infty W_\infty{\cong}B{\otimes}_{\mathbb{R}}W \tag{4.4}$$

We conclude this section by noting that:

Proposition 4.9. *Let A be a C^∞-algebra. If I is an ideal of the ring defined by forgetting the extra C^∞-algebra structure of A and $\phi{\in}C^\infty(\mathbb{R}^n)$ then:*

$$x_i - y_i{\in}I{\Rightarrow}\phi(\vec{x}) - \phi(\vec{y}){\in}I$$

Proof. By Equation 4.1:

$$\phi(\vec{x}) - \phi(\vec{y}) = \sum_{i=1}^{n}(x_i - y_i)\psi_i(\vec{x})$$

for some smooth ψ_i. But the right hand side is clearly an element of I. □

This means that if we are given a C^∞-algebra structure on A then the quotient ring A/I can be given a well-defined C^∞-algebra structure such that the natural map:

$$A \twoheadrightarrow A/I$$

is a C^∞-algebra homomorphism. We say that a C^∞ algebra is *finitely presented/generated* when the underlying ring is. Note that by Proposition 4.8 $W_\infty \cong C^\infty(\mathbb{R}) \otimes_{\mathbb{R}} W$ is finitely presented (as in Section 3.2) if and only if its underlying ring is.

4.2 Germ-determined C^∞-algebras

4.2.1 Definitions and First Properties

Given a point $p \in \mathbb{R}^n$ we can define an equivalence relation on the functions $\mathbb{R}^n \to \mathbb{R}$ by letting $\phi \sim \psi$ when they coincide on some open neighbourhood of p. We call this quotient ring the *ring of germs* at p. Formally (and more generally), let $J(p)$ be the ideal of functions of $C^\infty(M)$ that vanish in some open neighbourhood of p. Then

$$C^\infty(M)/J(p)$$

is the algebra of germs. By Proposition 4.9 we have a unique C^∞-algebra structure on $C^\infty(M)/J(p)$.

Definition 4.10. Let $\phi_i : M \to \mathbb{R}$ be smooth where M is a smooth manifold. Then *supp*(ϕ_i) is the closure of the set of points in V for which ϕ_i is non-zero. We say that a collection of smooth ϕ_i is *locally finite* if the *supp*(ϕ_i) are locally finite.

Definition 4.11. A *locally finite partition of unity* ϕ_i subordinating an open cover $(U_i)_i$ of a smooth manifold M are smooth maps

$$\phi_i : U_i \to \mathbb{R}$$

such that

(i) *supp*$(\phi_i) \subseteq U_i$.

(ii) for every point $p \in M$ we have that there exists a neighbourhood W of p such that W meets only finitely many of the *supp*(ϕ_i).

(iii) $\sum_i \phi_i = 1$.

We will use locally finite partitions of unity to extend functions $b_i : U_i \to \mathbb{R}$ to a function $b : \bigcup_i U_i = V : \to \mathbb{R}$. As an example, although the function $\phi_i b_i$ is only defined on U_i it can naturally be extended to a smooth map $V \to \mathbb{R}$ because $\phi_i = 0$ outside U_i. To aggregate the b_i we will often form the sum $\sum_i \phi_i b_i$ which has domain V.

Therefore we will need following theorem from the general theory of Differential Geometry:

Theorem 4.12. *Let X be a paracompact Hausdorff space. Then for each open cover of X there is a locally finite partition of unity.*

Proof. See Theorem 1 of Chapter 3 of [Arhangel'skii, 1995]. □

Definition 4.13. Let A be a C^∞-algebra and X be a set of elements of A. Then $A[X^{-1}]$ is the universal solution in Alg_∞ of adjoining inverses of all $x \in X$ to A.

Definition 4.14. Σ_p (defined in context of a C^∞-algebra A) is the set of all elements $a \in A$ such that $a(p) \neq 0$.

Proposition 4.15.

$$C^\infty(M)/J(p) \cong C^\infty(M)[\Sigma_p^{-1}]$$

Proof. We need to show that:

(i) all $f \in C^\infty(M)$ such that $f(p) \neq 0$ are invertible in $C^\infty(M)/J(p)$.

(ii) all $g \in C^\infty(M)$ that vanish on a neighbourhood of p are zero in $C^\infty(M)[\Sigma_p^{-1}]$.

Then we will be done by the universal properties of the two objects and the fact that if there is a unique arrow $A \to B$ and a unique arrow $B \to A$ then $A \cong B$.

(i) Let f be such that $f(p) \neq 0$. Then there exists a neighbourhood U of p such that $\frac{1}{f}$ is well-defined. Let ϕ, ψ be a (clearly locally finite) partition of unity subordinate to $U, \{p\}^C$. Note that $\psi \in J(p)$. Then we extend $\frac{1}{f}$ to $\phi \frac{1}{f}$. Intuitively we expect this to be the inverse of f in $C^\infty(M)/J(p)$ because in this quotient ring functions coinciding on a neighbourhood of p are the same. Indeed:

$$f \phi \frac{1}{f} = \phi = 1 - \psi \equiv 1$$

(ii) Let g vanish on the neighbourhood U of p. Again let ϕ, ψ be a partition of unity subordinate to $U, \{p\}^C$. Note that $\phi \in \Sigma_p$ since $\phi(p) = 1$. Then $\phi \cdot g \equiv 0$ on M. But ϕ is invertible in $C^\infty(M)[\Sigma_p^{-1}]$ and so $g \equiv 0$.

□

Definition 4.16. A *point* of a C^∞-algebra A is a C^∞-algebra homomorphism

$$A \to \mathbb{R}$$

The following theorem from the general theory of differential geometry justifies this definition:

Theorem 4.17. *Let M be a manifold. If $p : C^\infty(M) \to \mathbb{R}$ is an $\mathbb{R}$-algebra map then there exists an unique point $P \in M$ such that p is evaluation at P. That is:*

$$p(f) = f(P)$$

for all $f \in C^\infty(M)$

Proof. See Corollary 3.3 in [Kriegl et al., 1989]. □

Because of Proposition 4.15 the following definition legitimately generalises the notion of germ algebra:

Definition 4.18. For *any* C^∞-algebra A the *germ algebra* at a point $p : A \to \mathbb{R}$ is the C^∞-algebra $A[\Sigma_p^{-1}]$ where $\Sigma_p = \{a \in A | p(a) \neq 0\}$.

Note that for C^∞-algebras of the form C^∞(M) for some smooth manifold M we recover the germ algebra described at the beginning of this section.

Notation: Let A be a C^∞-algebra. For $a \in A$ we write a_p for the image of a under the natural map

$$A \twoheadrightarrow A[\Sigma_p^{-1}]$$

For $I \triangleleft A$ we write

$$I_p := \langle a_p : a \in I \rangle \triangleleft A_p$$

In words: $a_p \in I_p$ if a has the same germ at p as some element in I.

Definition 4.19. An ideal I of a C^∞-algebra A is *germ-determined* if

$$I = \hat{I} := \{a \in A | a_p \in I_p (\forall p : A \to \mathbb{R})\}$$

A C^∞-algebra is *germ-determined* if the zero ideal is germ-determined.

Remark 1. We have that $I \subseteq \hat{I}$ and

$$I = \hat{I} \iff (a_p \in I_p \Rightarrow a \in I)$$

In words: if (at every point p) a has the same germ as some element of I then $a \in I$.

Example 4.20. $C^\infty(\mathbb{R}^n)/I$ is germ-determined if I is germ-determined.

4.2.2 Finitely Presented Implies Germ-determined

Lemma 4.21. *Let $I \lhd C^\infty(M)$ and $b \in \hat{I}$. Then for each point p of $C^\infty(M)$:*

$$b = x^{(p)} + y^{(p)}$$

where $x^{(p)} \in I$ and $y^{(p)} \in J(p)$ (and so $y^{(p)}$ vanishes on some neighbourhood $V^{(p)}$ of p).

Proof. $b \in \hat{I}$ implies that for each point p of A we have $b_p \in I_p$. That is we can find $x^{(p)} \in I$ such that $x^{(p)}_p = b_p$. So on some neighbourhood $V^{(p)}$ of p

$$b - x^{(p)}$$

vanishes. □

Lemma 4.22. *Let U be an open set in a smooth manifold M and V be a closed set contained in U. Then there exists a smooth function $\psi : M \to \mathbb{R}$ such that ψ is zero outside U and 1 inside V.*

Proof. Consider a locally finite partition of unity ϕ, ψ subordinating the open cover U, V^C. Then ψ is as required. □

Recall that because of Proposition 4.15 we can conflate points $p \in M$ with points of $C^\infty(M)$ (that is C^∞-algebra homomorphisms $p : C^\infty(M) \to \mathbb{R}$).

Theorem 4.23. *If*

$$K \lhd C^\infty(M)$$

is germ-determined and $a \in C^\infty(M)$ then $\langle K, a \rangle = I$ is germ-determined.

Proof. Let $b \in C^\infty(M)$ such that $b_p \in I_p$ for all points p in $C^\infty(M)$. We will show that $b \in I$.

By Lemma 4.21 we have that

$$b = x^{(p)} + y^{(p)}$$

where $x^{(p)} \in I$ and $y^{(p)}$ vanishes on a open neighbourhood $V^{(p)}$ of p.

Take a locally finite refinement $(U_i)_i$ of the cover given by all the $V^{(p)}$ (which is possible since M is paracompact). Then let ϕ_i be a locally finite partition of unity subordinating $(U_i)_i$. This means that for $U_i \subseteq V^{(p_i)}$

$$\phi_i b = \phi_i x^{(p_i)} + \phi_i y^{(p_i)} = \phi_i x^{(p_i)} \in I$$

and so $\phi_i b = k^{(p_i)} + l^{(p_i)} a$ for $l^{(p_i)}$ smooth and $k^{(p_i)} \in K$.

Now we would like to sum over i to regain b but we don't know that $l^{(p_i)}$ and $k^{(p_i)}$ are locally finite. (Local finiteness will ensure that for each $m \in M$

the sums $\sum_i l^{(p_i)}(m)$ and $\sum_i k^{(p_i)}(m)$ are only over a finite number of non-zero terms and so we can manipulate the summations more freely.) However, we may make them so by multiplying by a smooth function ψ_i (whose existence is assured by Lemma 4.22) which is zero outside U_i and 1 inside $supp(\phi_i)$. This means that the supports of $\psi_i m^{(p_i)}$ and $\psi_i k^{(p_i)}$ are contained in U_i and $(U_i)_i$ is locally finite. So we have:

$$\phi_i b = \psi_i k^{(p_i)} + \psi_i m^{(p_i)} a$$

since $\phi_i = 0$ outside U_i. Thus:

$$b = \sum_i \phi_i b = \sum_i \psi_i k^{(p_i)} + \sum_i \psi_i l^{(p_i)} a$$

So we will be done if we can show that $\sum_i \psi_i k^{(p_i)} \in K$. But $k^{(p_i)} \in K$ so:

$$(k^{(p_i)})_p = (\psi_i k^{(p_i)})_p \in K_p$$

and therefore

$$\left(\sum_i \psi_i k^{(p_i)} \right)_p \in K_p$$

for all p. But K is germ-determined so we do indeed have $\sum_i \psi_i k^{(p_i)} \in K$. □

Corollary 4.24. *All finitely presented C^∞-algebras are germ-determined.*

4.3 Characteristic Functions and Whitney Embedding Theorem

In this section we will sketch the proof that $C^\infty(M)$ is finitely presented for all smooth manifolds M.

Lemma 4.25. *Let U be an open subset of a smooth manifold M. Then there exists a* characteristic function $\chi_U : M \to \mathbb{R}$ *which has*

$$\{m \in M | \chi_U(m) \neq 0\} = U$$

Proof. First consider the case that U is an open ball within a single coordinate chart with $\vec{0}$ as the centre of the ball. Then define χ_U to be 0 on U^C and on U:

$$\chi_U(\vec{x}) = exp\left(\frac{1}{|x|^2 - R^2} \right)$$

where R is the radius of the ball (which is well-defined since there is a single coordinate chart containing U).

For general U write $U = \bigcup_i U_i$ and define

$$\chi_U(\vec{x}) = \sum_i \chi_{U_i}(\vec{x}) \cdot a_i$$

where $a_i \in \mathbb{R}$ are chosen so that χ_U is smooth. These can be found because χ_{U_i} have compact support. □

Lemma 4.26.
$$C^\infty(U)\cong C^\infty(\mathbb{R}^n)[\chi_U^{-1}]$$
for all open $U \subseteq \mathbb{R}^n$*.*

Proof. Let

$$\hat{U} = \{(x_1, x_2, ..., x_n, y)\in\mathbb{R}^n | y\chi_U(x_1, x_2, ..., x_n) = 1\}$$

Then
$$C^\infty(U)\cong C^\infty(\hat{U})\cong C^\infty(\mathbb{R}^n)[\chi_U^{-1}]$$
by the universal property of $C^\infty(\mathbb{R}^n)[\chi_U^{-1}]$. □

Proposition 4.27. $C^\infty(U)$ *is finitely presented for all open* $U \subseteq \mathbb{R}^n$*.*

Proof. Let $\hat{U}$ be as above. Then

$$C^\infty(U)\cong C^\infty(\mathbb{R}^{n+1})/I$$

where I is the ideal of all $\phi : \mathbb{R}^{n+1}\to\mathbb{R}$ that vanish on $\hat{U}$. But by Hadamard's Lemma if $\vec{x} = (x_1, x_2, ..., x_{n+1})$ then:

$$\phi(\vec{x}) = (x_{n+1}\cdot\chi_U(x_1, x_2, ..., x_n) - 1)\cdot\psi$$

for some smooth $\psi : \mathbb{R}^{2n}\to\mathbb{R}^n$. Therefore I is in fact principally generated and $C^\infty(U)$ is finitely presented. □

In order to conclude that $C^\infty(M)$ is finitely presented for all smooth manifolds M we use the following result from the general theory of differential geometry:

Theorem 4.28. *(Whitney) Every manifold* M *embeds in* $\mathbb{R}^N$ *for some* $N\in\mathbb{N}$*.*

Proof. See Section 8 of Chapter 1 of [Guillemin and Pollack, 1974]. □

Therefore M can identified with a retract of an open subset $U \subseteq \mathbb{R}^N$. Since $C^\infty(M)\cong C^\infty(U)$ we conclude from Proposition 4.27 that:

Corollary 4.29. $C^\infty(M)$ *is finitely presented for all smooth manifolds* M*.*

5 Models for Synthetic Differential Geometry

Now we construct a model of a smooth topos.

Following [Johnstone, 2006] we say that a topos $\mathcal{E}$ is a well-adapted model of Synthetic Differential Geometry if:

I. There is a full embedding $\iota : Man \to \mathcal{E}$.

II. ι sends open covers in Man to jointly epimorphic families in $\mathcal{E}$.

III. The Kock-Lawvere axiom holds when $(\mathcal{E}, \iota(\mathbb{R}))$ is a taken to be the lined topos in the definition of smooth topos (Definition 3.14).

IV. A certain type of pullback (*transversal* pullbacks) are preserved by ι.

We shall show that if $\mathcal{C}$ is any well-adapted category (see Definition 5.1) then for

$$\mathcal{E} = Sh(\mathcal{C}^{op}, D)$$

and ι the composite:

$$Man \xrightarrow{C^\infty(-)} \mathcal{C}^{op} \xrightarrow{y} Sh(\mathcal{C}^{op}, D)$$

the conditions I, II and III are satisfied. The verification of IV will not be given in this essay. Here D is the Debuc coverage (see section 5.3), $Sh(\mathcal{C}^{op}, D)$ is the topos of sheaves on the site $(\mathcal{C}^{op}, D)$ and y is the Yoneda embedding.

This section is based on F1.3 of Volume 3 of [Johnstone, 2006] and the verification of IV may be found there. Chapters III.7 and III.8 of [Kock, 2006] were also used.

Finally we shall show that Weil spectra are tiny in the presheaf topos $[Alg_\infty, Set]$ and refer the reader to the end of Chapter III.8 of [Kock, 2006] for the work required to extend this result to show that the first condition in the definition of smooth topos (that Weil spectra are tiny) is satisfied in $Sh(\mathcal{C}^{op}, D)$.

5.1 Well Adapted Categories

Definition 5.1. A *well-adapted category* $\mathcal{C}$ is a full subcategory of Alg_∞ such that:

(i) $\mathcal{C}$ contains all C^∞-algebras of the form $C^\infty(M)$ for M a smooth manifold.

(ii) For any $A \in \mathcal{C}$ and Weil algebra W over $\mathbb{R}$ we have that $A \otimes_{\mathbb{R}} W \in \mathcal{C}$.

(iii) If $A \in \mathcal{C}$ and $a \in A$ then $A[a^{-1}] \in \mathcal{C}$.

(iv) All objects of $\mathcal{C}$ are finitely generated and germ-determined.

Well-adapted categories include:

(a) The category of all C^∞-algebras of the form $C^\infty(M)\otimes_{\mathbb{R}}W$ for M a smooth manifold.

(b) The category of finitely presented C^∞-algebras.

(c) The category of finitely generated and germ-determined C^∞-algebras.

We have proved enough to verify that (b) is a well-adapted category:

(i) Corollary 4.29 is that all C^∞-algebras of the form $C^\infty(M)$ are finitely presented.

(ii) $A\otimes_{\mathbb{R}}W$ is finitely presented if A is.

(iii) $A[a^{-1}]$ is finitely presented if A is.

(iv) Corollary 4.24 is that all finitely presented C^∞-algebras are germ-determined.

In the following sections (4.1.1-4.1.3) we will work with an arbitrary well-adapted category $\mathcal{C}$.

5.2 The Embedding $Man\hookrightarrow\mathcal{C}^{op}$

Recall Theorem 5.2 from the classical theory of differential geometry:

Theorem 5.2. *Let M be a manifold. If $p : C^\infty(M)\to\mathbb{R}$ is an $\mathbb{R}$-algebra map then there exists an unique point $P\in M$ such that p is evaluation at P. That is:*

$$p(f) = f(P)$$

for all $f\in C^\infty(M)$

Proof. See Corollary 3.3 in [Kriegl et al., 1989]. □

Since the Yoneda embedding is full and faithful, in order to show that the functor

$$y\circ C^\infty(-) : Man\to Sh(\mathcal{C}^{op}, D)$$

is a full and faithful it will suffice to show that

$$C^\infty(-) : Man\to\mathcal{C}^{op}$$

is. First we prove:

Theorem 5.3.

$$C^\infty(-) : Man\to Alg_{\mathbb{R}}^{op}$$

is full and faithful.

Proof. $C^\infty(-)$ *is faithful:* Let $h \neq k : M \to N$. Then there exists a point $x \in M$ such that $h(x) \neq k(x)$ and we can find $f : N \to \mathbb{R}$ taking different values on $h(x)$ and $k(x)$. Thus $f \circ h \neq f \circ k$. Now considering f as an element of $C^\infty(N)$ we see that $C^\infty(h) \neq C^\infty(k)$.

$C^\infty(-)$ *is full:* Let $\phi : C^\infty(N) \to C^\infty(M)$. Now for every $y \in M$ we may compose with the map $C^\infty(y)$ that is evaluation at y to get:

$$C^\infty(N) \xrightarrow{\ \phi\ } C^\infty(M) \xrightarrow{\ C^\infty(y)\ } \mathbb{R}$$

But by Theorem 5.2 we see that $C^\infty(y) \circ \phi$ is the same as evaluation at some point $k(y) \in N$. Thus $k(y)$ defines a map $k : M \to N$. By construction for $f \in C^\infty(N)$:

$$\phi(f)(y) = f(k(y)) = (f \circ k)(y)$$

and so $C^\infty(k) = \phi$.

□

Now we use the fact that the forgetful functor $Alg_\infty^{op} \to Alg_\mathbb{R}^{op}$ is faithful to get that $C^\infty(-) : Man \to Alg_\infty$ is full and faithful. Then we appeal to condition (i) of the definition of well-adapted category to conclude that $C^\infty(-) : Man \to \mathcal{C}^{op}$ is full and faithful for any well-adapted category $\mathcal{C}$.

5.3 The Debuc Coverage

Now we use the identification

$$C^\infty(U) = C^\infty(M)[\chi_U^{-1}]$$

(c.f. Lemma 4.26 where U is an open subset of a smooth manifold M and χ_U is any function $M \to \mathbb{R}$ for which $\chi_U(m)$ is non-zero precisely when $m \in U$) to transfer the notion of an open cover from Man to $\mathcal{C}^{op}$ for a well-adapted category $\mathcal{C}$.

Definition 5.4. The *Debuc coverage* D on $\mathcal{C}^{op}$ (where $\mathcal{C}$ is a well-adapted category) has covering families (of each $A \in \mathcal{C}$) given by all families

$$(f_i : A \to A[a_i^{-1}])_{i \in I}$$

in $\mathcal{C}$ such that every point $p : A \to \mathbb{R}$ factors through some f_i. Equivalently: for each point p there is at least one i with $p(a_i) \neq 0$. We say that the family $(f_i)_i$ is *a Debuc cover of A*.

The existence of a smooth characteristic function χ_U for every open set $U \subseteq M$ (Lemma 4.25) implies that $C^\infty(-)$ takes open covers in Man to Debuc covers in $\mathcal{C}^{op}$: first we express

$$C^\infty(U) = C^\infty(M)[\chi_U^{-1}]$$

Thus every inclusion $U \hookrightarrow M$ is sent to a restriction

$$C^{\infty}(M) \to C^{\infty}(U) = C^{\infty}(M)[\chi_U^{-1}]$$

by $C^{\infty}(-)$. Therefore an open cover $(U_{\alpha})_{\alpha}$ of M is sent to a Debuc cover.

Before we show that the Debuc covering is subcanonical we prove:

Lemma 5.5. *Let $A \in Alg_{\infty}$ be finitely generated and germ-determined. Then $a \in A$ is invertible if and only if for all points $p : A \to \mathbb{R}$ we have that $p(a) \neq 0$.*

Proof. Since A is finitely generated and germ-determined we have that

$$A = C^{\infty}(\mathbb{R}^n)/I$$

where I is germ-determined. Choose f such that $\bar{f} = a$ where $\bar{f}$ is the image of f under the natural map $C^{\infty}(\mathbb{R}^n) \twoheadrightarrow C^{\infty}(\mathbb{R}^n)/I$. Then

$$A/(a) \cong C^{\infty}(\mathbb{R}^n)/\langle I, f \rangle$$

and so by Proposition 4.23 we have that $A/(a)$ is germ-determined. But by $p(a) \neq 0$ for all $p : A \to \mathbb{R}$ we see that $A/(a)$ has no points and thus $(a) = A$. □

Corollary 5.6. *Let V be an open subset of $\mathbb{R}$ and let*

$$\pi : C^{\infty}(\mathbb{R}^n) \twoheadrightarrow A$$

be an epimorphism. If every point p of A has $p \circ \pi$ as a point of $C^{\infty}(V)$ then π factors through the restriction

$$C^{\infty}(\mathbb{R}^n) \to C^{\infty}(V)$$

Proof. If every point p of A has $p \circ \pi$ as a point of $C^{\infty}(V)$, then $p(\pi(\chi_V)) \neq 0$. But then $\pi(\chi_V)$ is invertible in A by the previous Lemma and (by Lemma 4.26) we have that

$$C^{\infty}(V) = C^{\infty}(\mathbb{R}^n)[\chi_V^{-1}]$$

and we get the required factorisation. □

Theorem 5.7. *The Debuc coverage on $\mathcal{C}^{op}$ is subcanonical.*

Proof. Since every representable functor preserves limits it suffices to show that for all $A \in \mathcal{C}$

$$A \longrightarrow \prod_{i=1}^{n} A[a_i^{-1}] \Longrightarrow \prod_{1 \leq i,j \leq n} A[a_i^{-1}] \times_A A[a_i^{-1}] \tag{5.1}$$

is an equaliser in Alg_∞, where $A[a_i^{-1}]\times_A A[a_j^{-1}]$ is the pushout:

$$\begin{array}{ccc} A & \xrightarrow{\ f_i\ } & A[a_i^{-1}] \\ {\scriptstyle f_j}\downarrow & & \downarrow \\ A[a_j^{-1}] & \dashrightarrow & A[a_i^{-1}]\times_A A[a_j^{-1}] \end{array}$$

and $(f_i : A \to A[a_i^{-1}])_{i\in I}$ a Debuc cover.

Now, A is finitely generated so we can find $\pi' : C^\infty(\mathbb{R}^m) \twoheadrightarrow A$. Choose U_i open subsets of $\mathbb{R}^n$ such that:

$$\begin{array}{ccc} C^\infty(\mathbb{R}^n) & \longrightarrow & C^\infty(U_i) \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi_i} \\ A & \xrightarrow{\ f_i\ } & A[a_i^{-1}] \end{array}$$

is a pushout. To show that Diagram 5.1 is an equaliser we must pick $\bar{b_i} \in A[a_i^{-1}]$ such that $(\bar{b_1}, \bar{b_2}, ..., \bar{b_n})$ is equalised by the parallel arrows in 5.1 and show that $(\bar{b_1}, \bar{b_2}, ..., \bar{b_n})$ factors through $(f_1, f_2, ..., f_n)$.

Let $\bigcup_i U_i = V$. Then every point p of A factors through some f_i (by definition of Debuc cover) and so after precomposition with π' is a point of $C^\infty(U_i)$. In particular it is a point of $C^\infty(V)$. So by Corollary 5.6 there exists a π such that $\pi' = \pi \circ rest_V$ where $rest_V : C^\infty(\mathbb{R}^n) \to C^\infty(V)$ is the restriction. Choose $b_i \in C^\infty(U_i)$ such that $\pi_i b_i = \bar{b_i}$. We will extend b_i to a map $V \to \mathbb{R}$ by using a locally finite partition of unity ϕ_i subordinating $(U_i)_i$. Firstly, $\phi_i b_i$ extends to a map $V \to \mathbb{R}$ and so $b := \sum_i \phi_i b_i \in C^\infty(V)$.

Now fix a point $p : U_i \to \mathbb{R}$. Choose a neighbourhood W of p such that W only meets finitely many $supp(\phi_i)$ (which is possible because $(\phi_i)_i$ is locally finite). Next choose an open set $W' \subseteq W$ such that p is a point of all the $supp(\phi_i)$ that meet W'. Let the set of i such that $supp(\phi_i)$ meets W' be I. Then:

$$\begin{aligned} (f_i \pi b)_p &= (\pi b)_p \\ &= (\pi \sum_{i\in I} \phi_i b_i)_p \\ &= (\pi \sum_{i\in I} \phi_i b_i)_p \end{aligned}$$

By definition of W'.

Let ι_i, ι_j be the inclusions of $A[a_i^{-1}], A[a_j^{-1}]$ respectively into $A[a_i^{-1}]\times_A A[a_j^{-1}]$. Now since the b_i are equalised by the two parallel lines in 5.1, we have that $\iota_i(b_i) = \iota_j(b_j)$ for all i, j. Now let p be a point of A that factors through $A[a_i^{-1}]$ and $A[a_j^{-1}]$. Then localising: $(b_j)_p = (b_i)_p$. Therefore since we defined I so that p is a point of all the $supp(\phi_i)$ that meet W:

$$\begin{aligned}(f_i\pi b)_p &= \sum_{i\in I}(\pi\phi_i)_p(\pi_i b_i)_p \\ &= \sum_{i\in I}(\pi\phi_i)(\pi_j b_j)_p = (\pi_j b_j)_p\end{aligned}$$

However, $A[a_i^{-1}]$ is germ-determined so:

$$f_i(\pi b) = \pi b_j = \bar{b_j}$$

and we have a factorisation of $(\bar{b_1}, \bar{b_2}, ..., \bar{b_n})$ through $(f_1, f_2, ..., f_n)$. □

5.4 The Kock-Lawvere Axiom

Consider the site $(\mathcal{C}^{op}, D)$ for a well-adapted category $\mathcal{C}$ and D the Debuc coverage on $\mathcal{C}^{op}$. Then since D is subcanonical the functor

$$R := Alg_\infty(C^\infty(\mathbb{R}), -)$$

which takes a C^∞-algebra to its underlying set is an object of $Sh(\mathcal{C}^{op}, D)$. We want to show that for all Weil algebras over $\mathbb{R}$:

$$R^{Spec_R(W)} \cong R\otimes_{\mathbb{R}} W$$

as R-algebras. To this end let $A\in\mathcal{C}^{op}$ and consider:

$$R^{Spec_R(W)}(A) = Nat\,(Alg_\infty(A, -)\times Spec_R(W), Alg_\infty(C^\infty(\mathbb{R}, -))$$

Recall (from Section 3.2) that $Spec_R(W)$ is defined as an equaliser of parallel arrows with domain and codomain R^d (where d is the dimension of the Weil algebra W). For example:

$$Spec_R(W) = \{r\in R | r^2 = 0\} \longrightarrow R \underset{0}{\overset{(-)^2}{\rightrightarrows}} R$$

But in $Sh(\mathcal{C}^{op}, D)$ this equaliser is computed pointwise. Therefore $Spec_R(W)(A)$ is the equaliser of parallel arrows in Set with domain and codomain RA^d (recall that RA is the underlying set of the C^∞-algebra A). In our example:

$$Spec_{RA}(W) = \{a\in RA | a^2 = 0\} \rightarrowtail RA \underset{0}{\overset{(-)^2}{\rightrightarrows}} RA$$

Thus $Spec_R(W)(A)$ is $Spec_{RA}(W)$. This means that $Spec_R(W)$ is represented by the canonical C^∞-algebra extending the algebra structure on W:

$$Spec_R(W) = Alg_\infty(W_\infty, -)$$

which is a sheaf since D is subcanonical. Therefore:

$$\begin{aligned} R^{Spec_R(W)}(A) &= Nat\,(Alg_\infty(A,-)\times Alg_\infty(W_\infty,-), Alg_\infty(C^\infty(\mathbb{R},-))) \\ &\cong Alg_\infty(C^\infty(\mathbb{R}), A+_\infty W_\infty) \\ &\cong R(A+_\infty W_\infty) \\ &\cong R(A\otimes_{\mathbb{R}} W) \\ &= (R\otimes_R W)(A) \end{aligned}$$

Using Proposition 4.8. So the R-algebra homomorphism

$$\alpha_W : R\otimes_{\mathbb{R}} W \rightarrow R^{Spec_R(W)} \tag{5.2}$$

between these functors defined by

$$w \mapsto (\phi \mapsto \phi(w))$$

is an isomorphism of R-algebras. Note that we verified (in a special case) that α_W is indeed an R-algebra homomorphism in Example 3.15. The general case is only more difficult in terms of notation.

5.5 Weil Spectra are Tiny

Lemma 5.8. *For any B in a category $\mathcal{A}$ the endo-functor*

$$(-)^{yB} : [\mathcal{A}^{op}, Set] \rightarrow [\mathcal{A}^{op}, Set]$$

preserves colimits. Here $y : \mathcal{A} \rightarrow [\mathcal{A}^{op}, Set]$ is the Yoneda embedding.

Proof. We will show that

$$(X^{yB})(C) \cong (colim_i(X_i^{yB}))(C)$$

for all $C \in \mathcal{A}$ and where $colim_i X_i = X$. Now,

$$\begin{aligned} (X^{yB})(C) &\cong [\mathcal{A}^{op}, Set](yC, X^{yB}) \\ &\cong [\mathcal{A}^{op}, Set](yC \times yB, colim_i X_i) \\ &\cong [\mathcal{A}^{op}, Set](y(C \times B), colim_i X_i) \\ &\cong (colim_i X_i)(C \times B) \\ &\cong colim_i(X_i(C \times B)) \\ &\cong colim_i [\mathcal{A}^{op}, Set](y(C \times B), X_i)) \\ &\cong colim_i [\mathcal{A}^{op}, Set](yC \times yB, X_i)) \\ &\cong colim_i [\mathcal{A}^{op}, Set](yC, X_i^{yB}) \\ &\cong colim_i (X_i^{yB}(C)) \\ &\cong (colim_i(X_i^{yB}))(C) \end{aligned}$$

□

Definition 5.9. The *category of elements* of $F \in [\mathcal{A}^{op}, Set]$ denoted by $\int_{\mathcal{A}} F$ has:

- pairs (x, c) as objects where $c \in \mathcal{A}$ and $x \in F(c)$.
- arrows

$$(x, c) \xrightarrow{f^*} (y, d)$$

  are arrows

$$d \xrightarrow{f} c$$

  in $\mathcal{A}$ such that $(Ff)y = x$. i.e. the 'restriction' of y along f is x.

Lemma 5.10. *A presheaf $F \in [\mathcal{A}^{op}, Set]$ is a colimit of a diagram of representable functors in $[\mathcal{A}^{op}, Set]$.*

Proof. There is a projection functor:

$$\textstyle\int_{\mathcal{A}} F \xrightarrow{\pi_{\mathcal{A}}} \mathcal{A}$$

which takes

$$(x, c) \xrightarrow{f^*} (y, d) \qquad \text{to} \qquad d \xrightarrow{f} c$$

This defines a diagram in $\mathcal{A}$. Composing with the Yoneda embedding $y : \mathcal{A} \hookrightarrow [\mathcal{A}^{op}, Set]$: gives us a diagram in $[\mathcal{A}^{op}, Set]$:

$$\textstyle\int_{\mathcal{A}} F \xrightarrow{\pi_{\mathcal{A}}} \mathcal{A} \xrightarrow{y} [\mathcal{A}^{op}, Set]$$

We will show that the colimit of this diagram is F.

F is the summit of a cone over $y \circ \pi_{\mathcal{A}}$: For each $(x, c) \in \int_{\mathcal{A}} F$, we must specify as a leg of the cone an arrow $yc \to F$ in $[\mathcal{A}^{op}, Set]$. The natural choice is the arrow that corresponds to $x \in F(c)$ under the bijection:

$$F(c) \cong [\mathcal{A}^{op}, Set](yc, F)$$

given by the Yoneda Lemma. The definition of the category of elements ensures that these legs describe a cone τ_1.

This cone is a limit: Let G be the summit of another cone τ_2 over $y \circ \pi_{\mathcal{A}}$. Then for each $(x, c) \in \int_{\mathcal{A}} F$ we have a leg $yc \to G$. But $yc \to G$ corresponds using the Yoneda Lemma to a point $x' \in G(c)$. Thus we define $F \to G$ pointwise by:

$$F(c) \to G(c)$$
$$x \mapsto x'$$

which (again by the definition of the category of elements) gives a factorisation of the cone τ_2 through the cone τ_1. □

Now that we know that $(-)^{yB} : [\mathcal{A}^{op}, Set] \to [\mathcal{A}^{op}, Set]$ preserves colimits and that any presheaf can be written as a colimit of representable presheaves, we have enough to specify a right adjoint to $(-)^{yB}$. This we will discover in a purely formal manner:

$$\begin{aligned}
[\mathcal{A}^{op}, Set](X^{yB}, Y) &\cong [\mathcal{A}^{op}, Set]((colim_i yX_i)^{yB}, Y) \\
&\cong [\mathcal{A}^{op}, Set](colim_i (yX_i)^{yB}, Y) \\
&\cong lim_i [\mathcal{A}^{op}, Set]((yX_i)^{yB}, Y) \\
&\cong lim_i [\mathcal{A}^{op}, Set](yX_i, [\mathcal{A}^{op}, Set](y(-)^{yB}, Y)) \\
&\cong [\mathcal{A}^{op}, Set](colim_i yX_i, [\mathcal{A}^{op}, Set](y(-)^{yB}, Y)) \\
&\cong [\mathcal{A}^{op}, Set](X, [\mathcal{A}^{op}, Set](y(-)^{yB}, Y))
\end{aligned}$$

So the right adjoint of $(-)^{yB}$ is

$$Y \mapsto [\mathcal{A}^{op}, Set](y(-)^{yB}, Y) \tag{5.3}$$

Now let $\mathcal{A} = \mathcal{C}^{op}$ in the above where $\mathcal{C}^{op}$ is a well-adapted category. Let D be the Debuc coverage on $\mathcal{C}^{op}$. Since D is subcanonical we have that $yW \in Sh(\mathcal{C}^{op}, D)$ and

$$(-)^{yW} : Sh(\mathcal{C}, D) \to Sh(\mathcal{C}, D)$$

It remains to show that the right adjoint to $(-)^{yW}$ in $[\mathcal{C}, Set]$ is an endofunctor on $Sh(\mathcal{C}^{op}, D)$ and for this we refer the reader to the end of Chapter III.8 in [Kock, 2006].

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